Graph Adjacency Matrix Automata

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Abstract

We define a *graph adjacency matrix automaton* (GAMA) to be a (finite) cellular automaton (CA) defined on an adjacency matrix of a simple graph in such a way that iterations of the automaton correspond to a simple graph. We will describe sufficient conditions on the update rule for a CA to be GAMA and give examples of such automata.

I. *Cellular Automata (CA)*

Definition 1 A *cellular automaton* (CA) is a lattice L in which each element has been assigned an element from a set A (the alphabet), together with a rule that updates the value assigned to a point in L according to the values at the neighboring points in the lattice.

A famous example of a CA is John Conway's Game of Life. Here, the lattice is 2-dimensional (integer coordinate points in the Cartesian plane), the alphabet is the set {0,1} (a cell with the value 0 is 'dead' whereas a cell with the value 1 is 'alive') and the update rule is defined as follows:

1.) Any live cell with fewer than two live neighbors dies, as if by loneliness.
2.) Any live cell with more than three live neighbors dies, as if by overcrowding.
3.) Any live cell with two or three live neighbors lives, unchanged, to the next generation.
4.) Any dead cell with exactly three live neighbors comes to life.

John Conway's Game of Life and advancements in computer technology have popularized mathematical research in cellular automata. Although CA in which the lattice corresponds to the adjacency matrix of a graph has yet to receive extensive study.

II. *GAMA*

An adjacency matrix of a simple graph can be viewed in a natural way as a 2-dimensional lattice of cells with the alphabet {0,1}. The motivation for this work began by asking whether we can define a cellular automaton update rule that always transforms an adjacency matrix of a simple graph to another adjacency matrix of a simple graph. This leads to the main definition of the paper.

Definition 2 *A graph adjacency matrix automaton* (GAMA) is a (finite) 2-dimensional CA where the lattice is an adjacency matrix of a simple graph with alphabet {0,1} and an update rule with the property that each state of the automaton corresponds to a simple graph.

Our initial investigations found examples of such rules, however there are certain conditions the local update rule of a CA must satisfy for it to define a GAMA. Namely, a finite 2-dimensional CA over {0,1} defines a GAMA only if its local update rule has the following characteristics:

1.) The main diagonal entries must have the value 0 at all time steps (simple graphs contain no loops).
2.) Symmetry must be maintained across the main diagonal (edges of simple graphs have no direction).

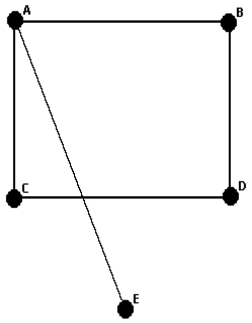
Conway's Game of Life rule can be applied to a finite lattice as long as we consider the lattice as on a 2-torus so that each cell has eight neighbors and this variation of Conway's Game of Life satisfies the above criteria. We state this as a formal proposition.

Proposition 1: Conway's Game of Life rule defines a GAMA.

Proof: The 8 neighbors of cell (i,j) have the same values as the corresponding neighbors for cell (j,i). Thus at each time step, the update applied to cell (i,j) is the same update applied to cell (j,i) so that the Game of Life update rule reserves the matrix symmetry. Also, since exactly 3 neighbors must be live to give birth and our matrix is symmetric, it is not possible to have an odd number of neighbors live for a diagonal that is initially 0 and hence the main diagonal entries remain zero at all time steps completing the proof.

Consider the following example.

Step 0:



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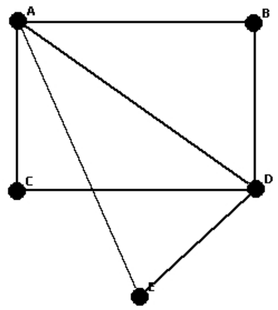
Step 1:



$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 1 & 1 & 0 & 1 \\ B & 1 & 0 & 0 & 1 & 0 \\ C & 1 & 0 & 0 & 1 & 0 \\ D & 0 & 1 & 1 & 0 & 0 \\ E & 1 & 0 & 0 & 0 & 0 \end{array} \rightarrow \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & 1 & 1 & 1 & 1 \\ B & 1 & 0 & 0 & 1 & 0 \\ C & 1 & 0 & 0 & 1 & 0 \\ D & 1 & 1 & 1 & 0 & 1 \\ E & 1 & 0 & 0 & 1 & 0 \end{array}$$

Remark: The update rule for the Game of Life defined as a GAMA is only well-defined on labeled graphs. That is, given two different labelings of the same graph, iterations of the Game of Life update rule on their respective adjacency matrices can produce adjacency matrices of non-isomorphic graphs. We leave it to the reader to construct her or his own examples. This motivates the following enhancement of the notion of GAMA.

Definition 3 A GAMA Γ is called *simple* if whenever A and A' are two adjacency matrices for isomorphic simple graphs, then $\Gamma^k(A) \cong \Gamma^k(A')$ for each k where $\Gamma^k(A)$ denotes the adjacency matrix of the graph created from A after k iterations of the update rule. We write $\Gamma(A)$ in place of $\Gamma^1(A)$.

Examples of simple GAMA are not difficult to construct. In fact, if we update the values in the (i,j) position using only values from the i^{th} row and j^{th} column (i.e. the (i,j) position update value depends solely on the values in the i^{th} row and j^{th} column and not the position of those values), the resulting GAMA is always simple. We state this as a theorem.

Theorem 1: If Γ is GAMA with the property that $\Gamma(A)_{ij}$ depends only on $\Gamma(A)_{kl}$ for k=i or l=j, then Γ is a simple GAMA.

Proof: Let Γ be defined as above and let G and G' be isomorphic graphs with adjacency matrices A and A' respectively. Since G and G' are isomorphic for A_{ij} there exists an $A'_{i'j'}$ such that row i' is a permutation of row i and column j' is a permutation of column j. Since the value for $\Gamma(A)_{ij}$ consists of operating on the entries in row i and column j regardless of the position of those values it must be the case that $\Gamma(A)_{ij} = \Gamma(A')_{i'j'}$. Therefore, $\Gamma(A) \cong \Gamma(A')$ and by induction $\Gamma^k(A) \cong \Gamma^k(A')$. Hence, Γ is a simple GAMA.

We illustrate with an example. Consider the following GAMA, Γ, with an update rule defined by

$$\Gamma(A)_{ij} = \left(\sum_{l=1}^{n} A_{il} + A_{lj}\right)(\bmod 2).$$

That is, the entry $\Gamma(A)_{ij}$ is the sum of the entries in row i and column j of A modulo 2. Consider the following example.

Step 0:

	A	B	C	D	E	F	G	H	I	J
A	0	1	0	1	0	0	0	1	0	1
B	1	0	0	1	0	1	1	1	1	0
C	0	0	0	1	1	0	1	1	0	1
D	1	1	1	0	1	1	0	0	1	0
E	0	0	1	1	0	0	1	1	1	1
F	0	1	0	1	0	0	1	1	0	0
G	0	1	1	0	1	1	0	1	0	0
H	1	1	1	0	1	1	1	0	1	0
I	0	1	0	1	1	0	0	1	0	0
J	1	0	1	0	1	0	0	0	0	0

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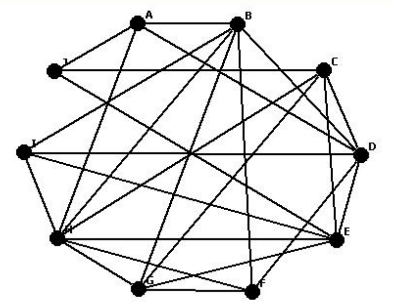
Step 1:

	A	B	C	D	E	F	G	H	I	J
A	0	0	1	0	0	0	1	1	0	1
B	0	0	1	0	0	0	1	1	0	1
C	1	1	0	1	1	1	0	0	1	0
D	0	0	1	0	0	0	1	1	0	1
E	0	0	1	0	0	0	1	1	0	1
F	0	0	1	0	0	0	1	1	0	1
G	1	1	0	1	1	1	0	0	1	0
H	1	1	0	1	1	1	0	0	1	0
I	0	0	1	0	0	0	1	1	0	1
J	1	1	1	1	1	1	0	0	1	0



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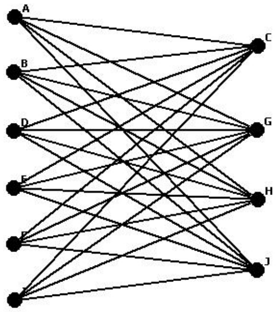

Step 1:

	A	B	C	D	E	F	G	H	I	J
A	0	0	1	0	0	0	1	1	0	1
B	0	0	1	0	0	0	1	1	0	1
C	1	1	0	1	1	1	0	0	1	0
D	0	0	1	0	0	0	1	1	0	1
E	0	0	1	0	0	0	1	1	0	1
F	0	0	1	0	0	0	1	1	0	1
G	1	1	0	1	1	1	0	0	1	0
H	1	1	0	1	1	1	0	0	1	0
I	0	0	1	0	0	0	1	1	0	1
J	1	1	1	1	1	1	0	0	1	0

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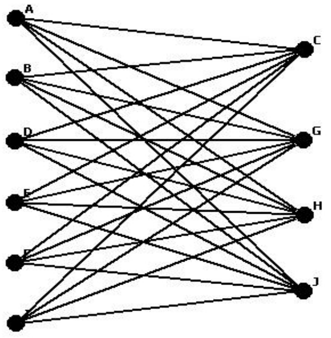
Step 2:

	A	B	C	D	E	F	G	H	I	J
A	0	0	0	0	0	0	0	0	0	0
B	0	0	0	0	0	0	0	0	0	0
C	0	0	0	0	0	0	0	0	0	0
D	0	0	0	0	0	0	0	0	0	0
E	0	0	0	0	0	0	0	0	0	0
F	0	0	0	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	0	0	0
H	0	0	0	0	0	0	0	0	0	0
I	0	0	0	0	0	0	0	0	0	0
J	0	0	0	0	0	0	0	0	0	0



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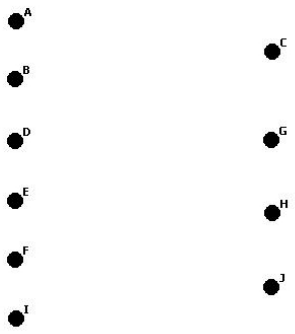

Proposition 2: Suppose a graph with adjacency matrix A, n x n, has p points of even degree and q points of odd degree. If p = 0 or q = 0 then Γ(A) is the null graph. Otherwise, Γ(A) is isomorphic to $K_{p,q}$.

Proof: Let A, n x n, be an adjacency matrix with p points of even degree and q points of odd degree. Now suppose p = 0 or q = 0, then $\Gamma(A)_{ij} = (\sum_{l=1}^{n} A_{il} + A_{lj})(\text{mod}\,2) = (a+b)\,(\text{mod 2})$, where a and b have the same parity. Thus $\Gamma(A)_{ij} = 0$ for all i, j. Hence Γ(A) is the null graph. Suppose both p and q are nonzero. For this portion of the proof we consider the update rule in graph theoretical terms. Note that $\Gamma(A)_{ij}$ is equal to the degree of the vertex corresponding to row i, we call this V_i, plus the degree of the vertex corresponding to column j, V_j, modulo 2. W.L.O.G consider row i in Γ(A), $\Gamma(A)_{ir}=1$ if V_r has the opposite parity of V_i and $\Gamma(A)_{ir} = 0$ if V_r has the same parity of V_i. Thus if V'_i is the vertex corresponding to row I in Γ(A), then V'_i is connected to each V'_r such that V_i and V_r have opposite parity. Thus Γ(A) corresponds to a graph that consists of two point sets, {all V with same parity as V_i} and {all V with opposite parity as V_i}, with each point connected to every point in the opposite point set. Hence, Γ(A) is isomorphic to $K_{p,q}$.

Sequences of iterations under this rule are easily described, but using our general class of rules, we can create examples with more complex behaviors.

For another example, consider the GAMA Γ with update rule defined by

$$b = \sum_{l=1}^{n} A_{il} + A_{lj}\ (\text{mod}\,4) \text{ and } \Gamma(A)_{ij} = \begin{cases} 1\ if\ A_{ij} = 0\ and\ b = 3 \\ 1\ if\ A_{ij} = 1\ and\ b = 2\ or\ 3 \\ 0\ otherwise \end{cases}.$$

This rule is similar to that in the Game of Life rule and the GAMA turns out to have properties similar to the Game of Life. Namely, after a finite number of time steps, there are graphs that evolve to the null graph, graphs that evolve to a stable state, and graphs that evolve into an oscillating phase [2 Conway].

We conclude by listing some questions for further research. What other general classes of non-trivial simple GAMA exist? Can we classify the characteristics of the update rules for simple GAMA? Does there exist a simple GAMA such that the number of distinct evolutions is equal to the number of distinct graphs of a certain order, i.e. evolutions partitioned by isomorphism classes of graphs of a certain order?

III. *References*

1. Bardzell, Michael. *The Evolution Homomorphism and Permutation Actions on Group Generated Cellular Automata. Complex Systems, Volume 15, Issue 2, 2004.*

2. Conway, John. As seen in *Martin Gardner's "Mathematical Games", Scientific American. Oct. 1970.*